Lec 1

($\mathbb{R}^3$ Simplified)

1.1 A vector quantity u is an ordered number triple $(x \;\; y \;\; z)$ in which x, y, z are real numbers. We write

$$u = (x \;\; y \;\; z)$$

x, y and z are the 1st, 2nd & 3rd components of u.

Fact: $(0 \;\; 0 \;\; 0)$ is called the null vector

$$0 = (0 \;\; 0 \;\; 0)$$

1.2 Norm of a vector:

$$\|u\| = \sqrt{x^2 + y^2 + z^2}$$

$\|u\|$ is called the 'magnitude' or length of u.

$\|0\| = 0$

Fact: If $\|u\|=1$, then u is called an unit vector.

Ex: $(\cos\theta \quad \sin\theta \quad 0)$ is an unit vector for any θ.

Ex $(\cos\theta\cos\phi \quad \cos\theta\sin\phi \quad \sin\theta)$ is an unit vector for any θ and ϕ.

1.3 Sum of two vectors:

$$u = (x_1 \quad y_1 \quad z_1)$$
$$v = (x_2 \quad y_2 \quad z_2)$$
$$u+v = (x_1+x_2 \quad y_1+y_2 \quad z_1+z_2)$$

Ex: $u = (1 \quad 2 \quad -5)$

$v = (-2 \quad 2 \quad 4)$

$$u+v = (1-2 \quad 2+2 \quad -5+4)$$
$$= (-1 \quad 4 \quad -1)$$

1.4 Scalar Multiplication:

$u = (x \quad y \quad z)$

λ is a scalar (real number).

$\lambda u = (\lambda x \quad \lambda y \quad \lambda z)$

Ex: $u = (1 \quad 2 \quad -5)$

$3u = (3\cdot 1 \quad 3\cdot 2 \quad 3\cdot(-5))$

$= (3 \quad 6 \quad -15)$

$v = (-2 \quad 2 \quad 4)$

$4v = (4\cdot(-2) \quad 4\cdot 2 \quad 4\cdot 4)$

$= (-8 \quad 8 \quad 16)$

$3u + 4v = (3-8 \quad 6+8 \quad -15+16)$

$= (-5 \quad 14 \quad 1)$

Fact:

$$\|\lambda u\| = |\lambda| \, \|u\|$$

1.5. Vector addition and triangle inequality

① $\|u+v\| \leq \|u\| + \|v\|$

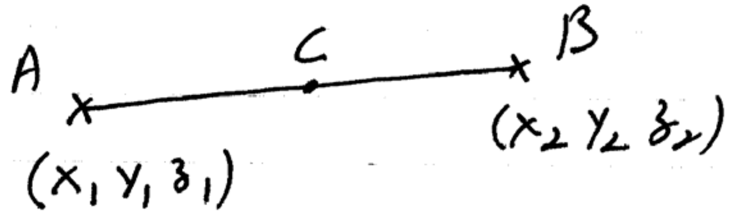
② $\|u-v\| \geqslant \big| \|u\| - \|v\| \big|$

1.6 The i, j and k :

$$i = (1 \quad 0 \quad 0)$$
$$j = (0 \quad 1 \quad 0)$$
$$k = (0 \quad 0 \quad 1)$$

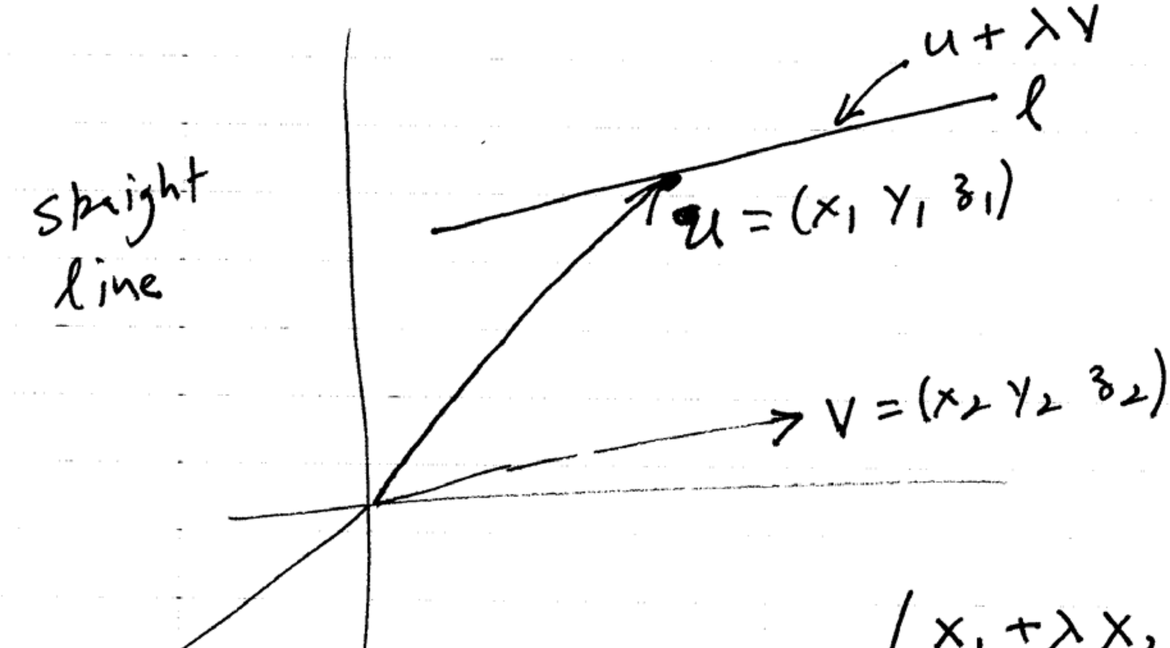
$$\begin{aligned} u &= (x \quad y \quad z) \\ &= x(1 \quad 0 \quad 0) + y(0 \quad 1 \quad 0) + z(0 \quad 0 \quad 1). \\ &= xi + yj + zk \end{aligned}$$

1.7 Simple geometrical application.



midpoint:

$$C: \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



$$u+\lambda v = \begin{pmatrix} x_1+\lambda x_2 \\ y_1+\lambda y_2 \\ z_1+\lambda z_2 \end{pmatrix}$$

Vector Eqn of line l

If $(x\ y\ z)$ is a point on l we have

$$x = x_1 + \lambda x_2$$

$$y = y_1 + \lambda y_2$$

$$z = z_1 + \lambda z_2$$

$$\boxed{\frac{x - x_1}{x_2} = \frac{y - y_1}{y_2} = \frac{z - z_1}{z_2}}$$

Equation of a straingt line

Ex: $\dfrac{x-5}{4} = \dfrac{y+2}{3} = -\dfrac{z-7}{8}$

$(5, -2, 7)$ is any point on the line

$(4\ 3\ -8)$ is a vector parallel to the line.

Ex

$$\frac{2x-3}{8} = \frac{3y+4}{9} = \frac{z-18}{20}$$

Write it as

$$\frac{x-3/2}{4} = \frac{y+4/3}{3} = \frac{z-18}{20}$$

$\left(\frac{3}{2}, -\frac{4}{3}, 18\right)$ is a point on the line

$(4 \quad 3 \quad 20)$ is a vector parallel to line l.

Note that these vectors are not unique. Can you find another vector pair??

Ex

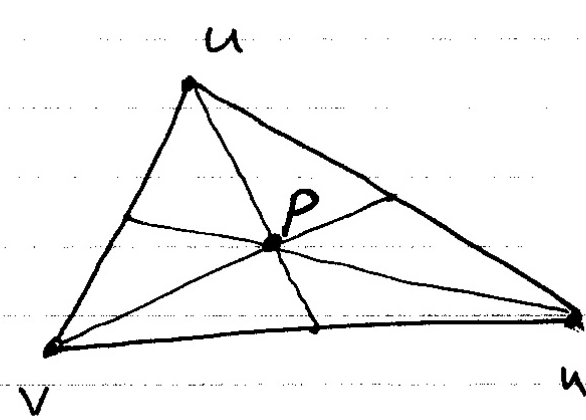
$$\frac{3x-1}{4} = \frac{2y+3}{2} = \frac{2-3z}{1} \quad \ell$$

(a) Find a unit vector u parallel to ℓ.

(b) Find the vector v on ℓ which is perpendicular to u.

(c) Are the vectors u, v unique??

Ex



Calculate the median vector P.

1.8 The Dot Product (Scalar product)

$$u \cdot v = \|u\| \cdot \|v\| \cos\theta . \quad (*)$$

fact:

Dot product is commutative

Dot product is distributive and linear

$$u \cdot v = v \cdot u .$$

$$u \cdot (v + w) = u \cdot v + u \cdot w .$$

$$u \cdot (\alpha v + \beta w) = \alpha u \cdot v + \beta u \cdot w .$$

Note that $(*)$ is not a ~~...~~ very useful definition because we dont know what θ is.

We write

$$u = (x_1 \ y_1 \ z_1) = x_1 i + y_1 j + z_1 k$$

$$v = (x_2 \ y_2 \ z_2) = x_2 i + y_2 j + z_2 k$$

~~$u \cdot v =$~~

$i = (1 \quad 0 \quad 0)$

$i \cdot i = \|i\| \, \|i\| \cos \theta^{=0}$

$= 1$

$j \cdot j = 1$

$k \cdot k = 1$

$i \cdot j = \|i\| \, \|j\| \cos 90^\circ = 0.$

etc.

It follows that

$$\boxed{u \cdot v = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

This is useful.

$$\boxed{\cos \theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \, \sqrt{x_2^2 + y_2^2 + z_2^2}}}$$

$0 \leq \theta \leq \pi$

1.9 $u = (x_1 \quad y_1 \quad z_1)$

$$u \cdot u = x_1^2 + y_1^2 + z_1^2$$

$$= \|u\|^2$$

Hence $\|u\| = \sqrt{u \cdot u}$.

If $u \cdot v = 0$ then we say that u and v are orthogonal or perpendicular.

Ex: $u = (1 \quad 2 \quad 3)$

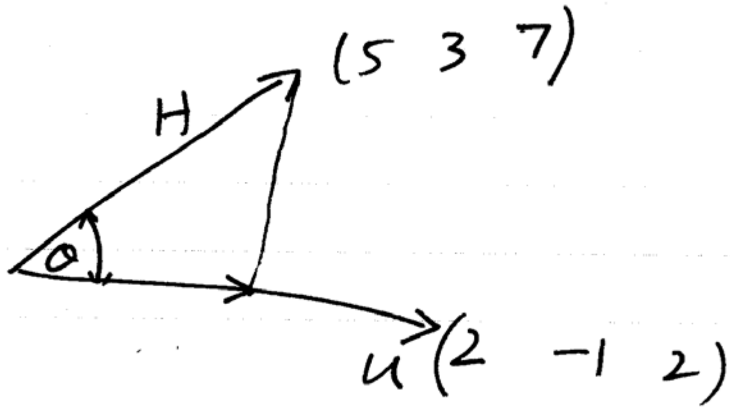
$v = (2 \quad -1 \quad -2)$

$\|u\| = \sqrt{14} \quad \|v\| = 3 \quad u \cdot v = -6$

$$\cos\theta = \frac{-6}{\sqrt{14}\,3} = -\frac{2}{\sqrt{14}}$$

$$\theta = 122.3^\circ$$

Ex: Find the strength of the magnetic field vector $H = (5 \quad 3 \quad 7)$ in the direction $(2 \quad -1 \quad 2)$



We need to project H in the direction of u.

$\|u\| = 3$

Define $\hat{u} = \dfrac{u}{\|u\|} = \left(\dfrac{2}{3} \quad -\dfrac{1}{3} \quad \dfrac{2}{3}\right)$

$\hat{u}$ is a unit vector in the direction u.

strength of H in the direction of u is

$$H \cdot \hat{u} = \|H\| \cos\theta = \frac{10}{3} - \frac{3}{3} + \frac{14}{3} = 7$$

1.10: Direction cosines :-

$$\cancel{u = (x \; y \; z)}$$

$$u = (a \quad b \quad c)$$

We want to calculate the angle θ_x between u and the x axis.

$$u \cdot i = \cos\theta_x \cdot \|u\| \cdot \|i\|$$

But $u \cdot i = a$

Hence

$$\cos\theta_x = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

Define $l = \cos\theta_x$ called the direction cosine of the angle between u and the x axis.

Likewise we have

$$m = \cos\theta_y = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \cos \theta_z = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

l, m, n are the three direction cosines.

Note that $l^2 + m^2 + n^2 = 1$.

Note also that

$$u = \|u\| \begin{pmatrix} l & m & n \end{pmatrix}$$

or $\|u\| (l\,i + m\,j + n\,k)$.

1.11: Cauchy-Schwarz Inequality

$$|u \cdot v| \leq \|u\| \, \|v\|.$$

An interesting calculation.

$$\|u + \lambda v\|^2 = (u + \lambda v) \cdot (u + \lambda v)$$

$$= u \cdot u + \lambda^2 v \cdot v + u \cdot \lambda v + \lambda v \cdot u$$

$$= \|u\|^2 + \lambda^2 \|v\|^2 + 2\lambda u \cdot v.$$

Since the above expression is true for any λ it follows that in particular it is true for

$$\lambda = -\frac{\|u\|^2}{u \cdot v}$$

This is a trick.

Thus .

$$\|u+\lambda v\|^2 =$$

$$\|u\|^2 + \frac{\|u\|^4}{(u\cdot v)^2}\|v\|^2 +$$

$$2\left(-\frac{\|u\|^2}{u\cdot v}\right)(u\cdot v)$$

$$= -\|u\|^2 + \frac{\|u\|^4\|v\|^2}{(u\cdot v)^2}$$

Since the l.h.s is always $\geqslant 0$

we have

$$\frac{\|u\|^4\|v\|^2}{(u\cdot v)^2} \geqslant \|u\|^2$$

Hence $(u\cdot v)^2 \leqslant \|u\|^2\|v\|^2$

Assuming $\|u\| \neq 0$.

and $|u\cdot v| \leqslant \|u\|\,\|v\|$.

If we set $\lambda = 1$ in the expression

$$\|u+\lambda v\|^2 = \|u\|^2 + \lambda^2 \|v\|^2 + 2\lambda\, u \cdot v.$$

we obtain.

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v.$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|$$

Hence

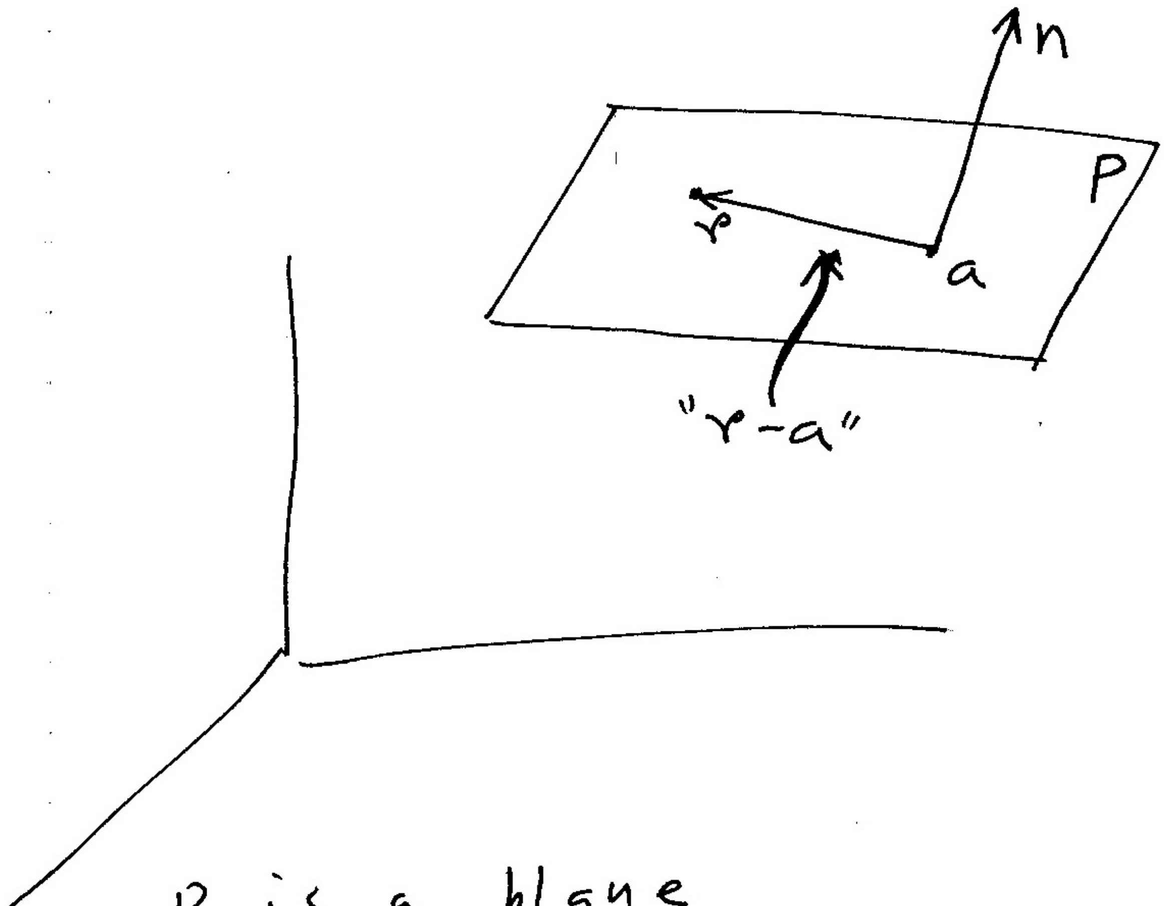
$$\|u+v\|^2 \leq \left[\|u\| + \|v\|\right]^2$$

or

$$\|u+v\| \leq \|u\| + \|v\|$$

(Triangle Inequality).

1.12 Equation of a plane



p is a plane

a is a vector on the plane

n is a vector perpendicular to the plane.

r is any other point on the plane.

$$a = (a_1 \quad a_2 \quad a_3)$$

$$n = (n_1 \quad n_2 \quad n_3)$$

$$r = (x \quad y \quad z)$$

we have

$$n \cdot (r - a) = 0$$

vector equation of the plane.

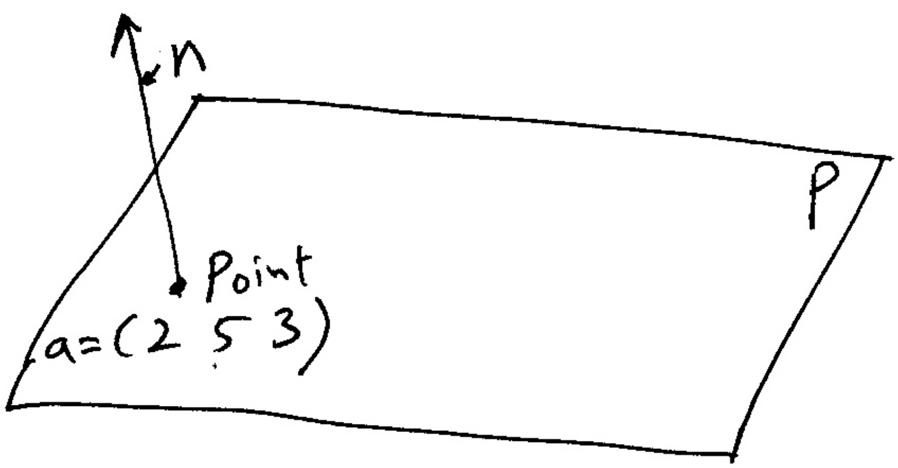
$$(n_1 \quad n_2 \quad n_3) \cdot (x - a_1 \quad y - a_2 \quad z - a_3) = 0$$

$$\boxed{n_1 x + n_2 y + n_3 z = n_1 a_1 + n_2 a_2 + n_3 a_3}.$$

"Cartesian Eqn of the plane"

Ex: consider a plane P through the point $(2 \quad 5 \quad 3)$ with normal.

$(3 \quad 2 \quad -7)$

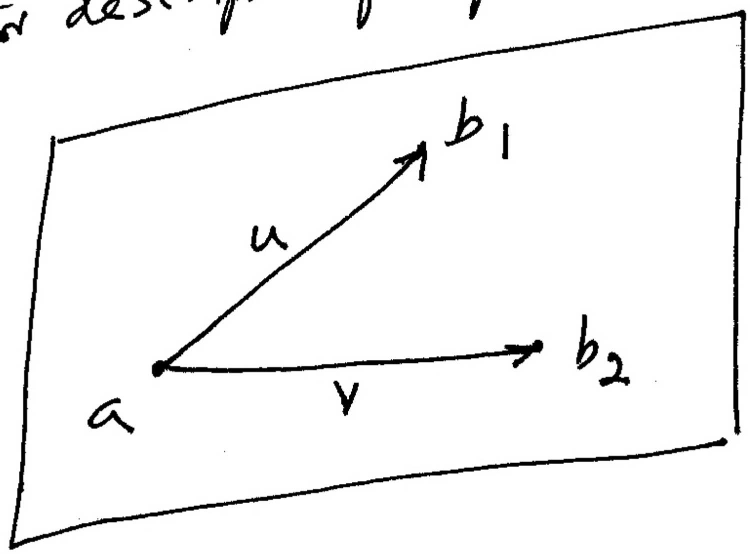


$n = (3 \quad 2 \quad -7)$

$3x + 2y - 7z = 6 + 10 - 21 = -5$

$$\boxed{3x + 2y - 7z = -5}$$

1.13 Vector description of a plane



$$a = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}. \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ 8/7 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

are 3 points on the plane P.

Define

$$u = b_1 - a = \begin{pmatrix} -1 \\ -5 \\ -\frac{13}{7} \end{pmatrix}$$

$$v = b_2 - a = \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$$

P is given by the set of points

$$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ -5 \\ -13/7 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$.

(compare this with page 5).

$$a + \lambda_1(b_1 - a) + \lambda_2(b_2 - a).$$

is the vector description of the plane